PARALLEL PLATE CAPACITOR

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Abstract. This document gives a short overview of some properties of a theoretical parallel plate capacitor. Its main purpose is to show that the surface charge does not distribute itself uniformly over the plates, to explain why, and to look at the consequences for the field, the potential and the capacitance.

1. Introduction

A parallel plate/plane capacitor is a fundamental configuration in electrostatics, consisting of two conducting plates of area A separated by a distance d (the gap usually being filled with a dielectric of permittivity $\epsilon = \epsilon_0\,\epsilon_r$). Under an applied voltage V, the plates carry equal and opposite charges $+Q$ and $-Q$, creating an electric field in the gap.

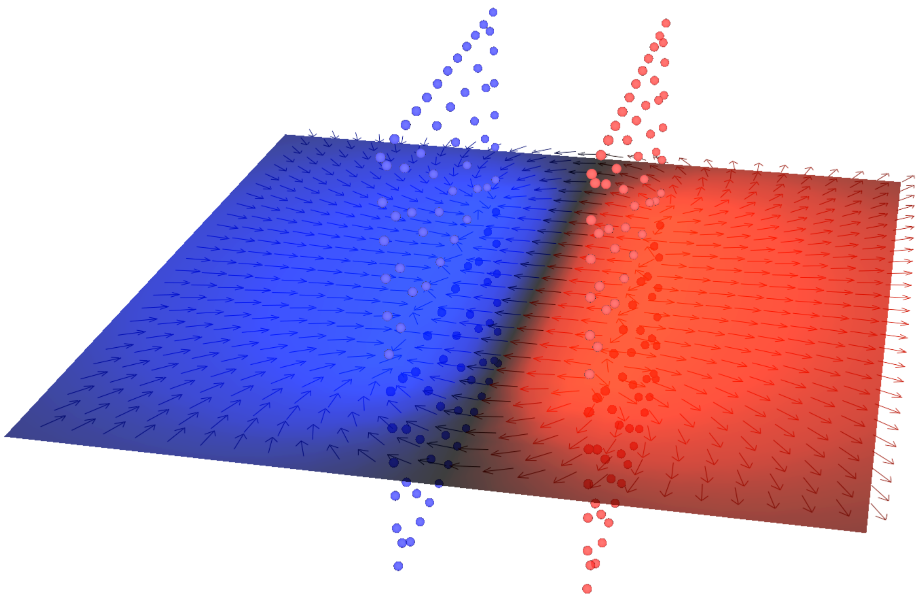


Figure 1. Test charges distribution in plane capacitor in Visualis Electromagnetism

In the idealized limit where the plates are infinitely large (or more realistically, where their linear size L satisfies $L \gg d$), the field in the central region is uniform, perpendicular to the plates, and equal to $E = V/d$; the capacitance is then

$$C_{\text{ideal}} = \frac{\epsilon A}{d}.$$

The rest of this note examines how, and how much, a real (finite) capacitor departs from this picture. The corrections are (literally) known as the *edge effects*.

2. Charge Distribution

2.1. **The distribution cannot be uniform.**

Each plate is a conductor, so in equilibrium it must be an *equipotential* : the tangential electric field on its surface vanishes, and no charge keeps moving along it. A uniform surface charge does not satisfy this condition for a finite plate ; a uniformly charged sheet has a higher potential at its centre than at its rim, which would drive charge outward. The charge therefore rearranges itself until the surface is at a single potential. Equivalently (Thomson's theorem), the equilibrium distribution is the one that *minimizes the electrostatic energy* for the given total charge confined to the conductor.

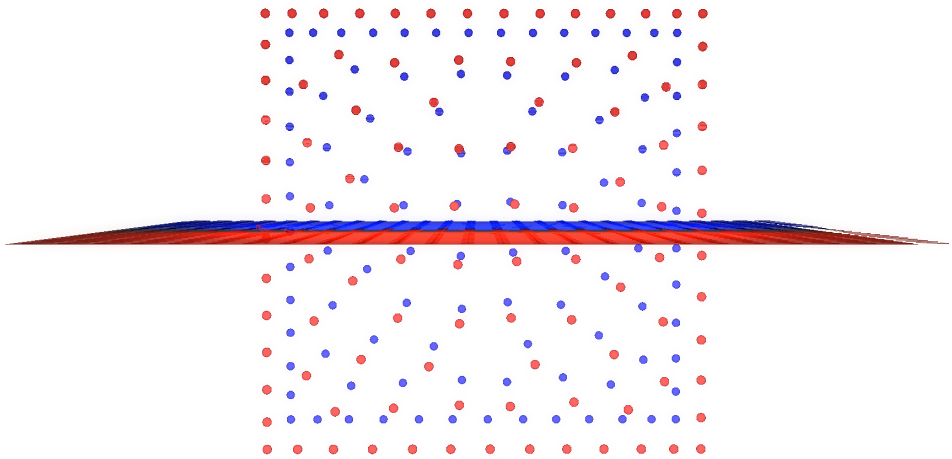


Figure 2. Charge distribution on plates, side view

The result is a surface charge density σ that is higher near the **edges and corners**, where the mutual repulsion of like charges is least opposed. For an idealized plate of zero thickness, σ actually *diverges* at the rim : if s is the distance to the edge,

$$\sigma(s) \sim s^{-1/2}, \qquad s \to 0.$$

This divergence is integrable, so the total charge on the plate remains finite. For a plate of finite thickness the density is large but finite at the rounded corners. Crucially, this non-uniformity is *localized* : it is confined to a band of width of order d around the perimeter. If $L \gg d$, the central part of the plate carries a nearly uniform charge and behaves essentially as in the ideal model.

2.2. **Effect on the field.**

It may seem paradoxical at first, but concentrating charge at the edges makes the field in the gap *more* uniform, not less. If the surface charge were artificially held uniform, the plates would no longer be equipotentials and the gap field (although strongest near the centre) would bow outward near the edges. By piling charge up toward the rim, the conductor raises the field there and keeps it straight and close to V/d over a larger central region. The price paid for this is twofold : the sharp rise (formally a divergence) of σ right at the edge, and the **fringing field** that bulges out of the gap and spills into the surrounding space just beyond the plates.

3. Electric Potential

The potential difference between the plates,

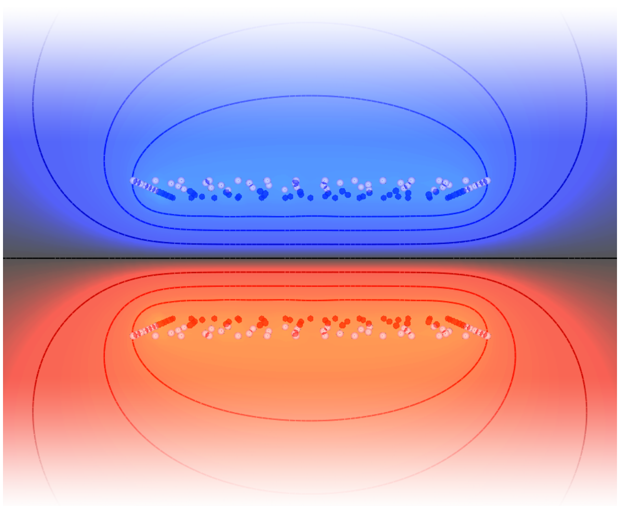
$$V = -\int \vec{E} \cdot d\vec{l},$$

reduces to the simple product $V = E\,d$ only in the central region, where $E \approx V/d$ is uniform. Near and beyond the edges the field is neither uniform nor purely perpendicular, so this elementary relation no longer holds pointwise.

In the dielectric gap, free of space charge, the potential obeys Laplace's equation

$$\nabla^2 V = 0,$$

(or Poisson's equation $\nabla^2 V = -\rho/\epsilon$ if a free space-charge density ρ is present), subject to the boundary conditions that V is constant on each conducting plate ($+V_0$ on one, $-V_0$ on the other or 0, depending on the reference) and decays at infinity. Solving this boundary value problem – by separation of variables, by conformal mapping in the two-dimensional cross section, or numerically (finite differences / FEM) – gives the true potential everywhere, including the fringing region, and hence the exact relation between Q and V.



4. Capacitance and Edge Corrections

Because the fringing field stores additional energy outside the nominal gap, the true capacitance is always *larger* than the ideal value :

$$C = \frac{\epsilon A}{d}\,\big(1 + \delta\big), \qquad \delta > 0.$$

The correction is an edge effect : it is proportional to the **perimeter** of the plates (not their area), and the relative correction scales as $\delta \sim (d/L)\ln(L/d)$, so it vanishes in the limit $d/L \to 0$. For a circular capacitor of radius R with $R \gg d$, Kirchhoff's classic result reads approximately

$$C \;\approx\; \epsilon_0 \left[\frac{\pi R^2}{d} + R\Big(\ln \frac{16\pi R}{d} - 1 \Big) \right],$$

where the first term is $C_{\text{ideal}} = \epsilon_0 A/d$ and the second, $\propto R$ (the perimeter), is the edge contribution. The take-away is robust even if one does not trust the exact constants : the ideal term scales with *area*, the dominant correction with *perimeter*.

5. Charging / Discharging Time

5.1. Notes and overview.

Small notes about charging / discharging capacitor behaviour, although this part is not handled by Visualis Electromagnetism directly.

It is tempting to think that the non-uniform charge distribution complicates the charging transient. By itself, it does not. During quasi-static charging the charge keeps the same relative (edge-enhanced) pattern at every instant, merely scaled by $Q(t)$; this geometric fact is already fully encoded in the single number C. For an ideal lumped RC circuit the charge therefore follows :

$$Q(t) = CV\big(1 - e^{-t/RC}\big), \qquad I(t) = \frac{dQ}{dt} = \frac{V}{R}\, e^{-t/RC},$$

a strictly linear (exponential) response with the single time constant $\tau = RC$.

Genuine departures from this ideal behaviour come from effects the lumped model omits, not from the static charge profile :

- the finite **sheet resistance of the plates** themselves, which turns the device into a *distributed RC* line, so that regions far from the electrical contact charge with a delay – a non-exponential, position-dependent transient ;
- **dielectric relaxation / dielectric absorption**, i.e. a frequency-dependent permittivity $\epsilon(\omega)$, which produces the familiar slow “soakage” tail ;
- at high frequency, the breakdown of the quasi-static approximation (signal propagation across the plates, skin effect, parasitic inductance).

5.2. Real Measurements.

As a side note, I include some direct measurements (and graphs) taken on a (large) parallel-plate capacitor dating back to high school :

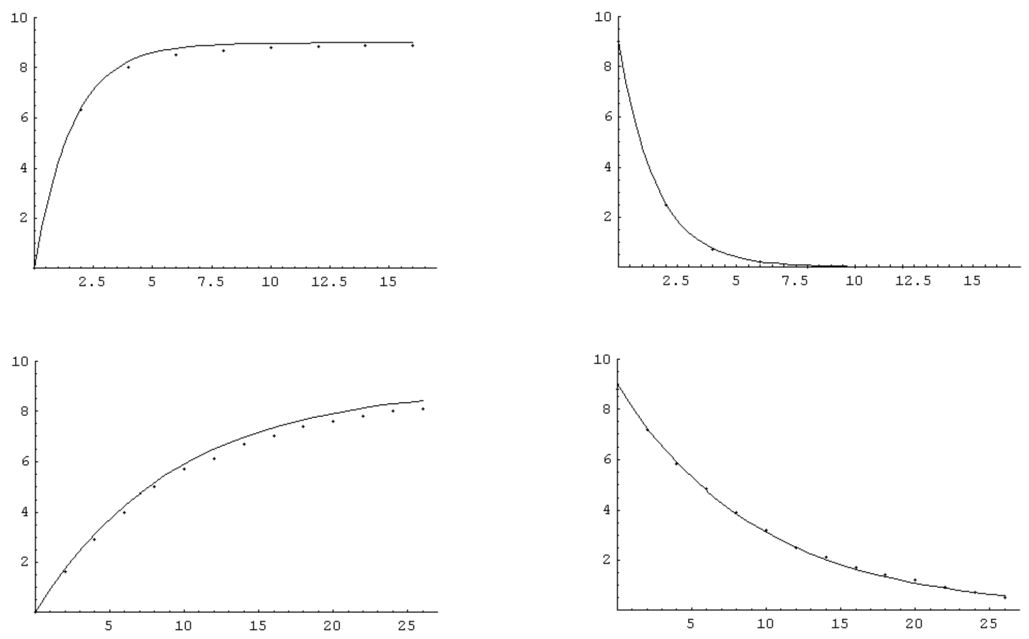


Figure 3. Experiment done at Nyon High School in 2002

With two different setups, these perfectly illustrate the exponential law we just discussed !

6. Numerical Simulation in Visualis Electromagnetism

In Visualis Electromagnetism the electric field and potential can be studied for either the uniform or the self-consistent (equilibrium) charge distribution. With the base Parallel Plate Capacitor demo scene, we can run the simulation and see the sample charges distribution, and the change in equipotential lines.

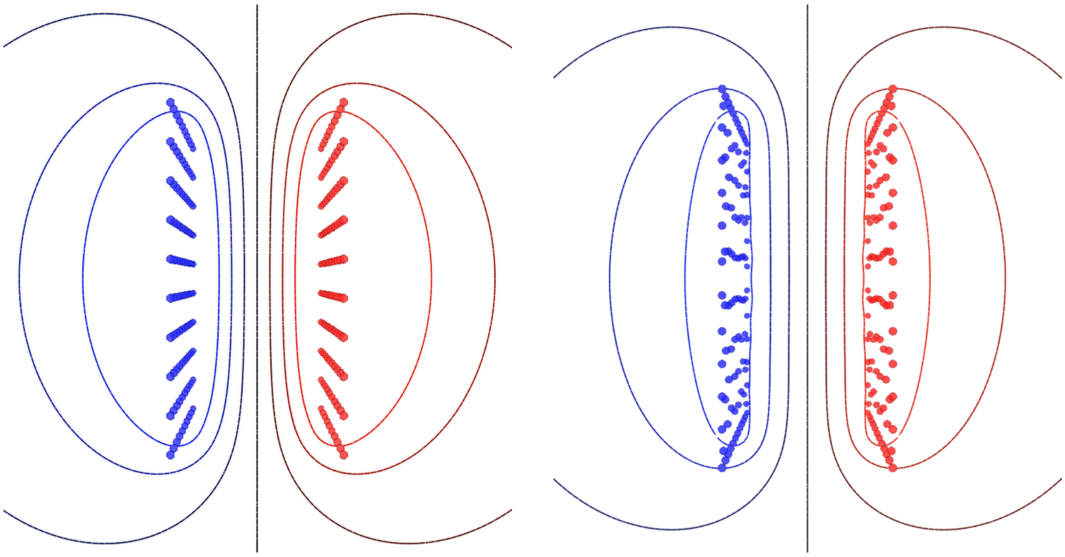


Figure 4. Top view of plane Capacitor, before and after charges free redistribution

The difference is very noticeable, after the simulation in the central region the equipotential lines straighten into a set of parallel lines/planes between the two plates, the signature of the uniform $E = V/d$ field, while near the rim they curve outward into the fringing region.

Letting theoretical test charges settle on a plate makes the edge crowding visible : they spread out and pile up toward the perimeter exactly as the $s^{-1/2}$ law predicts, rather than spacing themselves evenly. Comparing the imposed-uniform case with the relaxed equilibrium case is a direct way to *see* why the natural distribution is the one that keeps the plates equipotential.

References

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